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Nonlocal Conservation Laws and Supersymmetric Heisenberg Spin Chain

A. Ghose Choudhury[1] and A. Roy Chowdhury[1]

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The role of nonlocal conservation laws and the corresponding charges are analyzed in the supersymmetric Heisenberg spin chain. It is observed that such nonlocal charges generate a graded-Yangian type algebra, by using the properties of the monodromy matrix and classical r matrix.

1. INTRODUCTION

In recent years the role of nonlocal charges and conservation laws has been studied in detail (Bernard and Leclair, 1990) and it has been emphasized that these can be used to set up a machinery very similar to quantum inverse scattering (Faddeev, 1989). On the other hand, previous studies showed that such nonlocal conserved quantities could be generated out of some hidden symmetry transformations of the system (Dolan, 1981; Devchand and Fairlie, 1982; Bohr and Roy Chowdhury, 1985). It has been demonstrated that there is a close relation between quantum group structure and the effect of nonlocal charges on physical states (Zamoldochikov, 1978, 1979). Perhaps the most important utilization of nonlocal charges was that of Zamoldochikov (1978, 1979), who showed that the exact S-matrix could be determined by use of such quantities.

Here we study the nonlocal conserved quantities in the case of the supersymmetric Heisenberg spin chain (Makhanov and Pashaev, 1990) that the corresponding nonlocal charges generate a Yangian-type algebra (Cherednik, 1992; Le Clair and Smirnov, 1991) with gradation. The derivation is done through the use of the graded classical r-matrix (Semenov, 1983) and the corresponding monodromy matrix.

[1]High Energy Physics Division, Department of Physics, Jadavpur University, Calcutta-32, India.

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2. FORMULATION

The supersymmetric Heisenberg spin chain is governed by

$$S_t = [S, S_x]_x \tag{1}$$

where S is the superspin variable belonging to the lie algebra $uspl(2/1)$ and can be represented as

$$S = \begin{pmatrix} S_3 + S_4 & S^- & C_1^- \\ S^+ & -S_3 + S_4 & C_2^- \\ C_1^+ & C_2^+ & 2S_4 \end{pmatrix} \tag{2}$$

with $S^\pm = S_1 \pm iS_2$, $C_2^\pm = S_7 \pm is_8$, and $C_1^\pm = S_5 \pm iS_6$. Here (S_1, S_2, S_3, S_4) are the bosonic components and (S_5, S_6, S_7, S_8) are the fermionic ones. The spin operator S satisfies the constraint

$$S^2 = 3S - 2I \tag{3}$$

Let the generators T_i describe the lie algebra $uspl(2/1)$ and be comprised of both even (B_m) and odd (F_α) set, where $B_m \equiv \{T_1, T_2, T_3, T_4\}$ and $F_\alpha = \{T_5, T_6, T_7, T_8\}$. The grading of these generators is defined by $g(B_m) = 0$ and $g(F_\alpha) = 1$. Define the graded commutators

$$(T_\mu, T_\nu) \equiv T_\mu T_\nu - (-1)^{g(T_\mu)\cdot g(T_\nu)} T_\nu T_\mu \tag{4}$$

Then the super-Lie algebra is defined by

$$(T_\mu, T_\nu) = C^\omega_{\mu\nu} T_\omega \tag{5}$$

The structure constants satisfy

$$C^\omega_{\mu\nu} = -(-1)^{g(T_\mu)g(T_\nu)} C^\omega_{\nu\mu} \tag{6}$$

Define the metric tensor $g_{\mu\nu}$ by

$$g_{\mu\nu} = C^\sigma_{\omega\mu}(-1)^{g(T_\omega)} C^\omega_{\sigma\nu} \tag{7}$$

Then one can observe that

$$K = T_1^2 + T_2^2 + T_3^2 + T_4^2 + i[T_5T_6 - T_6T_5 + T_7T_8 - T_8T_7] \tag{8}$$

is a Casimir operator. The Lax pair pertaining to equation (1) can be written as

$$\left.\begin{aligned} \Psi_x &= U\Psi \\ \Psi_t &= V\Psi \end{aligned}\right\} \tag{9}$$

where

$$U = i\lambda S$$

$$V = i\lambda^2 S + \lambda[S, S_x] \tag{10}$$

The corresponding monodromy matrix is defined by

$$\frac{\partial T(x, y, \lambda)}{\partial x} = U(x, \lambda)T(x, y, \lambda) \tag{11}$$

along with the condition $T(x, x, \lambda) = 1$, so that the formal solution can be written as

$$\begin{aligned} T(x, y, \lambda) &= P \exp\left[\int_y^x U(z, \lambda)\, dz\right] \\ &= \mathbb{1} + (i\lambda) \int_y^x S(z)\, dz - \lambda^2 \int_y^x dz\, s(z) \int_y^z s(z)\, dz + \cdots \end{aligned} \tag{12}$$

By virtue of the equation of motion we observe that $\rho_1(z) = S(z)$ is the first conserved current, which is local, so we set

$$Q^0 = \int_{-\infty}^{\infty} \rho_1(z)\, dz \tag{13}$$

to be the first charge. Again from equation (12), we note that

$$\rho_2(z) = S(z) \int_y^z S(z')\, dz' \tag{13}$$

is the first nonlocal current. If we evaluate the time derivative of ρ_2 we get

$$\frac{\partial \rho_2}{\partial t} = -\frac{\partial}{\partial x}\left[iS(z) - i[S_z, S] \cdot \int_{-\infty}^{z} S(z')\, dz\right] \tag{15}$$

whence the first nonlocal charge is

$$Q^1 = \int_{-\infty}^{\infty} \rho_2(z)\, dz \tag{16}$$

We now rewrite the charges in terms of super-Lie algebra generators:

$$Q^0 = \int_{-\infty}^{\infty} dx\, S_a(x) T_a \tag{17}$$

so that

$$\{Q_a^0, Q_b^0\} = C_{abc} Q_c^0 \tag{18}$$

On the other hand,

$$\begin{aligned} Q^1 &= \int_{-\infty}^{\infty} s(z)\, dz \int_{-\infty}^{z} s(z')\, dz' \\ &= \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz'\, S_i(z) s_j(z) T_i T_j = \sum_{k=0}^{8} Q_k^1 T_k \end{aligned} \tag{19}$$

where

$$Q_0^1 = 1/2 \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz' [s_1(z)s_1(z') + s_2(z)s_2(z') + s_3(z)s_3(z')$$
$$- s_4(z')s_4(z') + is_5(z)s_6(z') - is_6(z)s_5(z') + is_7 s_8(z) - is_8(z)s_7(z')]$$

$$Q_1^1 = 1/2 \int_{-\infty}^{\infty} dz \int_{-\infty}^{z} dz' \Big[s_1(z)s_4(z') + s_4(z)s_1(z')$$
$$+ is_2(z)s_3(z') - is_3(z)s_2(z') + 1/2\, s_5(z)s_7(z')$$
$$+ 1/2\, s_7(z)s_5(z') + \frac{i}{2} s_5(z)s_8(z') - \frac{i}{2} s_8(z)s_5(z')$$
$$- \frac{i}{2} s_6(z)s_7(z') + \frac{i}{2} s_7(z)s_6(z') + 1/2\, s_6(z)s_8(z')$$
$$+ 1/2\, s_8(z)s_6(z') \Big] \tag{20}$$

with similar expressions for the other Q_j^1, whence by explicit computation

$$\{Q_a^0, Q_b^1\} = C_{abc} Q_c^1 \tag{21}$$

On the other hand, this algebra of the nonlocal supersymmetric charges may be understood from the similar properties of the associated nonlocal transformations. Let us go back to the lax equations (9) and (10) and rewrite the zero-curvature condition as

$$\partial_x A_0 - \partial_t A_1 = [A_0, A_1]$$

with

$$A_0 = i\lambda S = g^{-1} \partial_0 g$$
$$A_1 = i\lambda^2 S + \lambda[s, s_x] = g^{-1} \partial_1 g$$

where $g \in$ Lie group $USPL(2/1)$. Let us assume that the transformation of g takes the form

$$\delta_a^k g = -g\theta_a^k(x) \tag{23}$$

so that

$$\delta_a^k A_\mu(x) = (\theta_a^k, A_\mu) - \partial_\mu \theta_a^k$$

The parenthesis stands for the graded commutator. In practice

$$\theta_a(x, \lambda) = -\sum_{k=0}^{\infty} \lambda^k \theta_a^k = T(x, \lambda) T_a T^{-1}(x, \lambda) \tag{24}$$

One may now easily verify that

$$(\delta_a, \delta_b) = \{(\theta_a, \theta_b) + \delta_a \theta_b - \delta_b \theta_a\} \tag{25}$$

This formula will be used to evaluate the repeated commutator of the transformation. The algebraic structure so generated is given by the relation

$$(\theta_q(\lambda), \theta_b(\mu)) + \delta_a\theta_b(\mu) - \delta_b\theta_a(\lambda) = C_{abc}\frac{\lambda\theta_C(\lambda) - \mu\theta_c(\mu)}{\lambda - \mu} \tag{26}$$

Expanding in Laurent series, we can at once obtain

$$(\delta^0_a, \delta^0_b) = C_{abc}\delta^0_c \tag{27a}$$

$$(\delta^0_a, \delta^1_b) = C_{abc}\delta^1_c \tag{27b}$$

$$(\delta^0_a, (\delta^1_b, \delta^1_c)) - (\delta^1_a, (\delta^1_b, \delta^0_c)) = 0 \tag{27c}$$

$$((\delta^1_a, \delta^1_b), (\delta^0_c, \delta^1_d)) + (-1)^\sigma((\delta^1_c, \delta^1_d), (\delta^0_a, \delta^1_b)) = 0 \tag{27d}$$

$$\sigma = g(X_a)g(X_c) + g(X_b)g(X_d)$$

It is now quite evident that equations (27a) and (27b) correspond to (18) and (21), while the relation (27c) corresponds to

$$\{Q^0_a, \{Q^1_b, Q^1_c\}\} + \{Q^1_a, \{Q^1_b, Q^0_c\}\} = 0 \tag{28}$$

3. CLASSICAL *r*-MATRIX

Further properties of the nonlocal charges are deduced with the help of the classical r-matrix, which is closely related to the Casimir operator noted before. For the present case it is given by

$$r(\lambda, \mu) = \frac{\lambda\mu}{\lambda - \mu}\{T^1_1T^2_1 + T^1_2T^2_2 + T^1_3T^2_3 - T^1_4T^2_4 + T^1_5T^2_6 - T^1_6T^2_5 + T^1_7T^2_8 - T^1_8T^2_7\} \tag{29}$$

with $T^1_i = T_i \otimes 1$ and $T^2_i = 1 \otimes T_i$; it is also equal to

$$\frac{\lambda\mu}{2(\lambda - \mu)}[P_s - e_{ii} \otimes e_{ij}]$$

P_s stands for the supersymmetric permutation operator. Now referring back to equation (12), we can write

$$T_{ab}(\lambda) = P\left(\exp\left[i\lambda\int_{-\infty}^{\infty} s(y)\,dy\right]\right)_{ab} = \delta_{ab} + \sum_{n=0}^{\infty}\lambda^{n+1}t^{(n)}_{ab} \tag{30}$$

On the other hand, we know that

$$\{T(\lambda) \underset{,}{\otimes} T(\mu)\} = 2[r(\lambda, \mu), T(\lambda) \otimes T(\mu)] \tag{31}$$

where $\underset{,}{\otimes}$ denotes the super Poisson bracket and $\otimes$ denotes the graded direct product. So equating similar powers of λ, we get

$$\{t_{\alpha\beta}^{(n)}, t_{ab}^{(m)}\} = \Bigg[\delta_{ab} t_{\gamma\beta}^{(n+m)}(-1)^R - \delta_{\gamma\beta} t_{\alpha\delta}^{(n+m)}(-1)^S + \sum_{k=0}^{n-1} t_{\alpha\delta}^{(n-k-1)} t_{\gamma\beta}^{(m+n)}(-1)^R - \sum_{k=0}^{n-1} t_{\gamma\beta}^{(n-k-1)} t_{\alpha\delta}^{(m+k)}(-1)^S\Bigg]$$

$$R = g(\delta)g(\beta) + g(\gamma)[g(\alpha) + g(\delta)]$$

$$S = g(\alpha)[2g(\gamma) + g(\beta)] \tag{32}$$

This relation contains as a special case the algebra generated by the nonlocal charges discussed above and gives the most general commutation rule defining an example of a graded Yangian.

4. DISCUSSION

In the above analysis we have shown how the nonlocal supercharges close to form a super-Yangian type algebra. These relations are nothing but a replica of the composition laws derived earlier for the hidden symmetry transformations.

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